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THE CYCLE INTERSECTION MATRIX AND APPLICATIONS TO PLANAR GRAPHS AND DATA ANALYSIS FOR POSTSECONDARY MATHEMATICS EDUCATION

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THE CYCLE INTERSECTION MATRIX AND APPLICATIONS TO PLANAR
GRAPHS AND DATA ANALYSIS FOR POSTSECONDARY MATHEMATICS

EDUCATION

BY

CAITLIN R. PHIFER

A DISSERTATION SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

IN

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UNIVERSITY OF RHODE ISLAND

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DOCTOR OF PHILOSOPHY DISSERTATION
OF
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2014

ABSTRACT

Given a finite connected planar graph G with s finite faces, we define the *cycle-intersection matrix*, $C(G) = (c_{ij})$ to be a symmetric matrix of size $s \times s$ where c_{ii} is the length of the cycle which bounds finite face i , and c_{ij} is the negative of the number of common edges in the cycles bounding faces i and j for $i \neq j$. We will show that $\det C(G)$ equals the number of spanning trees in G . As an application, we compute the number of spanning trees of grid graphs via Chebychev polynomials. In addition, we show an interesting connection between the determinant of $C(G)$ to the Fibonacci sequence when G is a triangulation of an n -gon by non-overlapping diagonals.

We also apply methods from graph theory to the field of postsecondary mathematics education. We describe here a remediation program designed to help calculus students fill in the gaps in their precalculus knowledge. This program has provided us with a way to strengthen the quantitative skills of our students without requiring a separate course. The data collected are analyzed here and suggestions for program improvement are made.

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I would like to begin by thanking my major advisor, Dr. Woong Kook, for his patience and guidance from the time I began as a graduate student through the completion of this thesis. I am fortunate to have had an extraordinary teacher and mentor throughout my graduate career. I also consider myself fortunate to have had a second advisor, Dr. Nancy Eaton. Dr. Eaton has guided me through this program since the day I arrived at the University of Rhode Island. Her door was always open for advice on teaching, taking classes, research, job interviews, and much more.

I would like to give special thanks to Dr. Jessica Libertini for her guidance with my educational research. Without her, I may never have had the opportunity to help develop a remediation and assessment program. I would also like to thank Dr. Thomas Bella for providing me with valuable data I needed to assess this program.

I would also like to thank my other committee members, Drs. Norman Finizio, Joan Peckham, and Lisa DiPippo. Each of these members has offered friendly assistance and valuable suggestions on not only my work, but on academic life in general. Finally, I would like to thank my fellow graduate students, who have provided great support and friendship over the last five years.

DEDICATION

For my family and friends. Your support means the world.

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CHAPTER 1

The Cycle Intersection Matrix and Applications to Planar Graphs

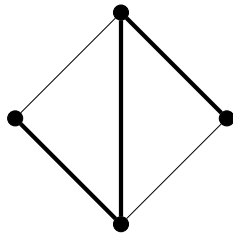
1.1 Introduction

Spanning trees are important substructures of a graph which are studied because of their simplicity and their relevance to various fields. In particular, the number of spanning trees in a graph is an interesting invariant. The problem of counting the number of spanning trees in a graph has been studied widely and has applications in computer science and network design. There exist various methods of finding this number, most of which rely on a matrix associated with the graph. Here, we will introduce a new matrix, called the Cycle Intersection matrix, and provide a new method to compute the tree number. We will see that this new method is more in line with Temperley's tree number formula than with the well-known Matrix-Tree theorem. Further, this new method reveals remarkably elegant results when applied to certain classes of planar graphs.

1.2 Background

The graphs we consider are finite, connected, and planar, meaning they consist of finitely many edges and vertices in a single component and can be drawn in the plane without crossing edges. We can assume that these graphs are *simple* so that there is at most one edge between every pair of vertices. Since our main focus is the number of spanning trees, we will also assume there are no loops. A *tree* is a graph which contains no cycles and a *spanning tree* is a subgraph of a graph G which uses all vertices of G . The number of spanning trees in a graph G is called the *tree number*, and will be denoted $\kappa(G)$. Spanning trees are sparse graphs, meaning they contain few edges. Because of this feature, they are used in many applications including computer networks, transportation routes, and water supply networks.

Example 1. *The bold edges denote a spanning tree:*



There are eight spanning trees in the graph from Example 1, as shown below.

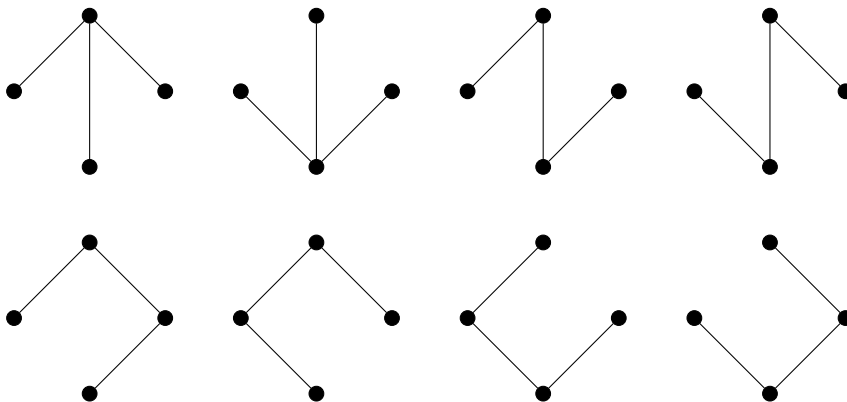
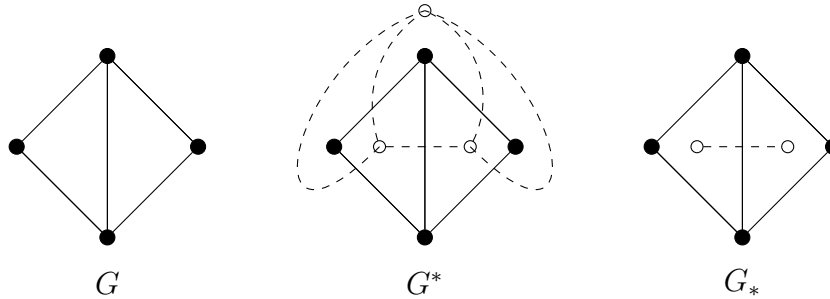


Figure 1. The eight spanning trees of the graph in Example 1.

When we draw a planar graph, we break up the plane into bounded regions, or *faces*. We consider the unbounded area outside of the graph to be a face and refer to it as the *infinite face*. For a planar graph, G , the *dual* is obtained by placing a vertex in each face of G and connecting two vertices each time their corresponding faces share an edge. We will use G^* to denote the dual of a planar graph G . Although this will be an important tool for us, we will also make use of the *weak dual*, denoted G_* , which is constructed the same way as the dual graph, but without placing a vertex in the infinite face.

Example 2. *The graph G and its dual and weak dual (dashed edges).*



1.2.1 Tree Number

Counting the number of spanning trees, or the tree number, in a graph is an important research area in combinatorics. The problem of calculating the tree number for a graph has been studied widely and has many applications in network analysis. In 1854, Gustav Kirchhoff showed that the number of spanning trees in a graph is equal to any cofactor of the Laplacian matrix of that graph [1]. Other methods for computing this number include use of the deletion-contraction formula [2] and Temperley's tree number formula [3]. More recently, in 1981, Bange, Barkauskas, and Slater [4] computed the number of spanning trees in triangulations using a reduction formula that makes use of the deletion-contraction formula. However, this method results in computing spanning trees of multiple graphs, some of which contain multiple edges. Here we will explore a new method of calculating this number using elementary ideas from linear algebra, which we will see is more straightforward than previous methods.

1.2.2 Tools for Computing the Tree Number

One of the main tools for computing $\kappa(G)$ is the Matrix-Tree theorem, proposed by Kirchhoff in 1854 [1]. This theorem relies on the Laplacian matrix, defined below.

Definition 1. The adjacency matrix, $A(G)$, of a graph G on n vertices is the $n \times n$ matrix given by

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } v_j \\ 0 & \text{if not.} \end{cases}$$

Notice that the adjacency matrix is a symmetric matrix whose rows and columns are indexed by the vertices of G . This matrix keeps track of the adjacencies among vertices and gives a complete description of the graph. Another important matrix related to a graph is the *diagonal matrix*, $D(G)$. The rows and columns are also indexed by the vertices of G and its entries are given by $d_{ii} = \deg v_i$ and $d_{ij} = 0$ for $i \neq j$. This matrix simply keeps track of the degree of each vertex. The $n \times n$ matrix $L(G) = D(G) - A(G)$ is called the *Laplacian matrix* of the graph G . An important property of $L(G)$ is that the sum of the entries in each row or column is zero, and hence, the matrix is not invertible. The theorem below is Kirkhoff's famous Matrix-Tree theorem.

Theorem 1. The number of spanning trees of a graph is equal to any cofactor of its Laplacian matrix.

Hence, if we denote by $L_0(G)$ the $(n-1) \times (n-1)$ submatrix of $L(G)$ obtained by removing an arbitrary row i and column i , the Matrix-Tree theorem implies that

$$\det L_0(G) = \kappa(G). \tag{1}$$

For the purpose of this work, we will always assume $L_0(G)$ is obtained from G by removing the last row and column, and we will refer to $L_0(G)$ as the *reduced Laplacian* of G .

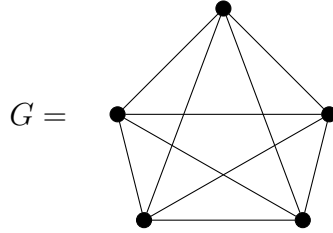
An analog of the Matrix-Tree theorem is Temperley's formula for $\kappa(G)$ [3]. We define the *augmented Laplacian* as $\mathcal{L}(G) = L(G) + J$ where J is the $n \times n$ matrix

whose entries are all 1. One reason we are interested in the augmented Laplacian matrix is that when G is connected, $\mathcal{L}(G)$ is invertible, unlike the Laplacian matrix. Temperley's formula states that

$$n^2 \kappa(G) = \det \mathcal{L}(G). \quad (2)$$

The next example illustrates the difference between the Matrix-Tree theorem and Temperley's formula in computing $\kappa(G)$ in the case of a complete graph.

Example 3. *We consider the complete graph on 5 vertices and compute $\kappa(G)$ in two ways.*



$$A(G) = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \quad D(G) = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

$$L(G) = \begin{pmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{pmatrix}$$

$$\text{By the Matrix-Tree theorem, } \kappa(G) = \det L_0(G) = \begin{vmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{vmatrix} = 125$$

In comparison, Temperley's formula is easier to work with in this case, since

$$\mathcal{L}(G) = \begin{pmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

so it is a quick calculation that $\kappa(G) = \frac{\det \mathcal{L}(G)}{n^2} = 125$.

In addition to the methods described above, one can also compute the number of spanning trees via the deletion-contraction formula [2]. This formula gives us a way to recursively compute the tree number by computing it for smaller graphs. Let $G - e$ denote the *deletion* of an arbitrary edge e from a graph and let G/e denote the *contraction* of edge e , which is obtained by identifying the two endpoints of e . The deletion-contraction formula tells us that

$$\kappa(G) = \kappa(G - e) + \kappa(G/e).$$

Although this formula allows us to compute the number of spanning trees in a graph with one less edge and one less vertex, it is not as efficient as the other methods discussed above.

Another tool for calculating $\kappa(G)$ was given by Kook in 2011 [5]. We define the *combinatorial Green's function*, \mathcal{G} of a graph G to be the inverse of the augmented Laplacian of G . That is, $\mathcal{G}(G) = \mathcal{L}(G)^{-1}$. Suppose that the endpoints of an edge e are the distinct vertices a and b . Then, with (g_{ij}) as the entries in \mathcal{G} , we have

$$g_{aa} + g_{bb} - g_{ab} - g_{ba} = \frac{\kappa(G/e)}{\kappa(G)}. \quad (3)$$

This result has applications in computing the resistance between two arbitrary nodes in a finite resistor network.

In the next section, we will propose a method for calculating $\kappa(G)$ using the Cycle Intersection matrix, which we will see is more in line with Temperley's formula since it does not require the deletion of rows and columns as in the Matrix-Tree theorem.

1.3 Cycle Intersection Matrix

Here, we introduce a new matrix and use it to calculate tree numbers for two families of graphs known as grid graphs and triangulations.

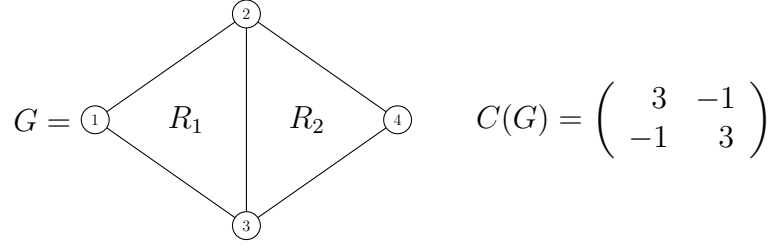
1.3.1 Preliminaries

This section consists of basic definitions and results concerning the Cycle Intersection matrix. We will show that the determinant of this matrix counts the number of spanning trees in a graph. Let $[n]$ denote the set $\{1, 2, \dots, n\}$. For a finite, connected, planar graph G , we create the Cycle Intersection matrix by first orienting the boundary of each finite face counterclockwise. Although the results below are independent of this orientation, it creates uniformity in the graph and imposes an advantageous structure on the matrix.

Definition 2. *Suppose G is a finite planar graph with s finite faces R_1, \dots, R_s . Denote the cycle bounding face R_i by C_i for $i \in [s]$. Let $E(C_i)$ denote the set of edges in cycle i . We define the cycle-intersection matrix of G , denoted $C(G)$ by $c_{ii} = |E(C_i)|$, and $c_{ij} = -|E(C_i) \cap E(C_j)|$ for $i \neq j$.*

It is clear that $C(G)$ is a symmetric matrix whose rows and columns are indexed by the cycles in G . The diagonal entries in the matrix keep track of the length of the cycle bounding each finite face, while the off-diagonal entries keep track of the number of edges two cycles have in common. This matrix is typically smaller than the Laplacian matrix and it is also invertible.

Example 4. *A graph and its cycle intersection matrix*



In comparison, the Laplacian matrix for the graph in Example 4 is of size 4×4 .

Lemma 2. *For a connected planar graph G , $\kappa(G) = \kappa(G^*)$.*

The proof of this lemma can be found in Exercise 5.23 [6]. Essentially, if T is a spanning tree in G , then the complement of the edges dual to T form a spanning tree in G^* . The main tool of this work is the following theorem, which connects $C(G)$ and $\kappa(G)$.

Theorem 3. *For G planar, $\det C(G) = \kappa(G)$.*

Proof. Suppose that G has s finite faces, labeled R_1, R_2, \dots, R_s , and that the infinite face is labeled R_{s+1} . As in the definition of $C(G)$, let C_i be the boundary of cycle i for $i \in [s]$. Consider the dual graph, G^* of G . We can construct the degree and adjacency matrices of G^* in the usual way, and denote these by $D(G^*)$ and $A(G^*)$. Then, $L(G^*)$ is the Laplacian matrix of G^* , and is of size $(s+1) \times (s+1)$. By construction, each vertex v_i in G^* has one edge for each edge in the face R_i of G , so $d_{ii}^* = |E(C_i)|$ for $i \in [s]$. Further, two vertices v_i and v_j in G^* are adjacent when their corresponding faces in G share an edge, so $a_{ij}^* = a_{ji}^* = |E(C_i) \cap E(C_j)|$ for $i, j \in [s]$. Since $L(G^*) = D(G^*) - A(G^*)$, this is precisely the definition of $C(G)$, so we have that $C(G) = L_0(G^*)$. Now, taking determinants and applying Equation 1 and Lemma 2, we have

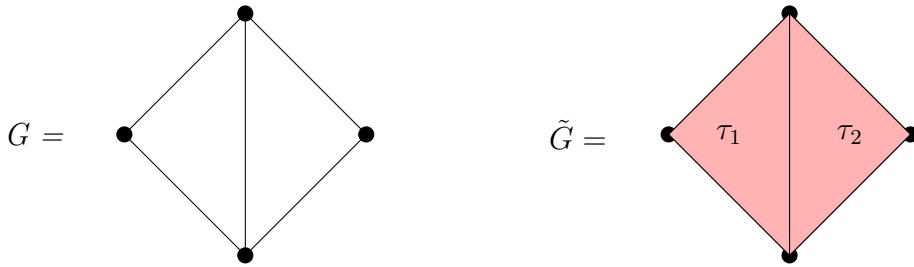
$$\det C(G) = \det L_0(G^*) = \kappa(G^*) = \kappa(G).$$

□

Theorem 3 tells us that we can calculate the tree number of a planar graph by simply calculating the determinant of its Cycle Intersection matrix. For instance, we can easily see that in Example 4, $\kappa(G) = 8$, which we observed in Figure 1.

We can extend Theorem 3 to any graph by using an acyclic augmentation. An *acyclic augmentation* of G is the acyclic 2-dimensional cell complex, \tilde{G} whose 1-skeleton is G .

Example 5. *An acyclic augmentation of a graph G :*



For a finite planar graph, \tilde{G} can be obtained by adding one 2-cell for each finite face of G . Since the combinatorial Laplacian in dimension 2 is the Cycle Intersection matrix, $\det C(G) = \kappa(G)$ for a general graph G (see Proposition 7 (3) [7]).

In the upcoming sections, we will use Theorem 3 to calculate the tree number for certain classes of graphs.

1.3.2 Grid Graphs

In this section, let $G = G_{m,n}$ denote the $m \times n$ *grid graph*, which is the cartesian product of the path graphs on m and n edges. This graph can easily be viewed as an $m \times n$ grid of squares. Although the tree number is independent of the labeling of the faces in the graph, labeling the finite faces by R_i for $1 \leq i \leq mn$ in

numerical order, from left to right, beginning at the top row results in a beneficial block structure in $C(G)$, as seen in Example 6 below.

Example 6. $G = G_{3,2}$ and its cycle intersection matrix:

$$G = \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline R_1 & R_2 & R_3 \\ \hline \bullet & \bullet & \bullet \\ \hline R_4 & R_5 & R_6 \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} \quad C(G) = \begin{pmatrix} 4 & -1 & 0 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 \\ -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 0 & -1 & 4 \end{pmatrix}$$

Note that $C(G)$ is of order $mn \times mn$ and, with this labeling, is a block tridiagonal matrix consisting of n^2 blocks, each of size $m \times m$. This block structure allows us to decompose $C(G)$ nicely using the Kronecker product.

Definition 3. The Kronecker product, $A \otimes B$, of matrices $A = (a_{ij})$ and $B = (b_{ij})$ is the block matrix whose (i, j) -th block is $a_{ij}B$.

Example 7. The Kronecker product of two 2×2 matrices is computed as follows:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \otimes \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} & a_{12} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \\ a_{21} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} & a_{22} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \end{pmatrix}$$

An important property of the Kronecker product is the identity

$$(A \otimes B)(C \otimes D) = AC \otimes BD. \quad (4)$$

Let I_n be the identity matrix of order $n \times n$. Lemma 4 will be useful for our main result regarding grid graphs. Its proof can be found in [8].

Lemma 4. *Let A and B be symmetric matrices of order $m \times m$ and $n \times n$, respectively. If $\{\lambda_i \mid i \in [m]\}$ and $\{\mu_j \mid j \in [n]\}$ are the eigenvalues of A and B , respectively, the eigenvalues of $A \otimes I_n + I_m \otimes B$ are $\{\lambda_i + \mu_j \mid i \in [m], j \in [n]\}$.*

Using Definition 3, we can decompose $C(G)$ into a sum of sparse matrices, each with a desirable structure.

Lemma 5. *Let U_n denote the $n \times n$ tridiagonal matrix with 1's on the upper and lower diagonals and 0's elsewhere. Then, the following identity holds for $C(G)$:*

$$C(G) = 4I_{mn} - (I_n \otimes U_m + U_n \otimes I_m).$$

Proof. Since every finite face of G is a square, all diagonal entries in $C(G)$ are 4, which gives us the first term $4I_{mn}$. Two finite faces intersect in at most one edge in G , and all are oriented counterclockwise, so all other entries in $C(G)$ are 0 or -1.

We can regard G as a collection n rows of m squares. There are two types of adjacencies among squares: one among the squares of a given row, and the other between the squares of row i and those of row $i + 1$ for $1 \leq i < n$. Note that there is no adjacency between row i and row j if $|i - j| \geq 2$.

The first type of adjacencies is given by $I_n \otimes U_m$ in the decomposition, which consists of n blocks of U_m on the main diagonal and 0 elsewhere. For each $i \in [n]$, the i -th block in $I_n \otimes U_m$ keeps track of adjacencies among the squares of row i . The second type of adjacencies is given by $U_n \otimes I_m$ in the decomposition, which is a block tridiagonal matrix with blocks of I_m on the upper and lower diagonals and 0 elsewhere. For each $1 \leq i < n$, the $(i, i + 1)$ -th block (also the $(i + 1, i)$ -th block) in $U_n \otimes I_m$ represents the adjacencies between the squares of row i and those of row $i + 1$. □

Example 8. With $G = G_{3,2}$ as in Example 6, we have

$$C(G) = 4I_6 - \begin{pmatrix} U_3 & 0 \\ 0 & U_3 \end{pmatrix} - \begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix}.$$

We now have our main result for the number of spanning trees in the $m \times n$ grid graph.

Theorem 6. The number of spanning trees of $G_{m,n}$ is given by

$$\kappa(G_{m,n}) = \prod_{(i,j) \in [m] \times [n]} 4 \sin^2 \left(\frac{i\pi}{2(m+1)} \right) + 4 \sin^2 \left(\frac{j\pi}{2(n+1)} \right).$$

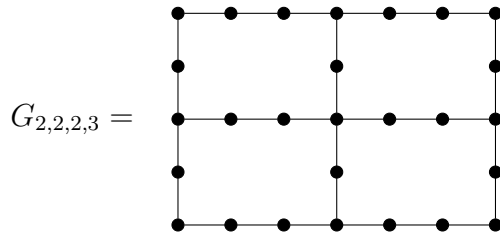
Proof. Note that U_n is the adjacency matrix of the path graph P_n with n vertices. It is well known that the eigenvalues of U_n are $U_n(\frac{x}{2})$, where $U_n(x)$ denotes the Chebyshev polynomial of the second kind [9]. Hence, the eigenvalues of U_n are $\lambda_i = 2 \cos(i\pi/(n+1))$ for $1 \leq i \leq n$ and the eigenvalues of $C(G)$ are

$$4 - 2 \cos(i\pi/(m+1)) - 2 \cos(j\pi/(n+1))$$

for all $i \in [m]$ and $j \in [n]$ by Lemma 4 and Lemma 5. The result follows from the half-angle formula for the sine function. \square

We can extend the result above to a more general grid graph. Let $G = G_{m,n,p,q}$ be the grid graph made up of m squares in the vertical direction and n squares in the horizontal direction where each square has p edges on its vertical sides and q edges on its horizontal sides, as in Example 9 below.

Example 9. $G_{2,2,2,3}$



Now, pU_n is the $n \times n$ tridiagonal matrix with p on the upper and lower diagonals and 0 elsewhere. The following lemma is a generalization of Lemma 5, and gives the decomposition of the matrix for this more general case.

Lemma 7. *The following identity holds for $C(G)$:*

$$C(G) = (2p + 2q)I_{mn} - (I_n \otimes pU_m + qU_n \otimes I_m).$$

The proof of this lemma is essentially the same as the proof of Lemma 5, and thus will be omitted. The next theorem is a generalization of Theorem 6. Its proof is similar to the proof of Theorem 6, and will also be omitted.

Theorem 8. *The number of spanning trees of $G_{m,n,p,q}$ is given by*

$$\kappa(G_{m,n,p,q}) = \prod_{(i,j) \in [m] \times [n]} 4p \sin^2 \left(\frac{i\pi}{2(m+1)} \right) + 4q \sin^2 \left(\frac{j\pi}{2(n+1)} \right).$$

From the number of spanning trees in the grid graph G_{mn} we have a formula for the spanning tree entropy [10], used by physicists, given by

$$\lim_{n,m \rightarrow \infty} \frac{1}{nm} \ln \kappa(G_{n,m}).$$

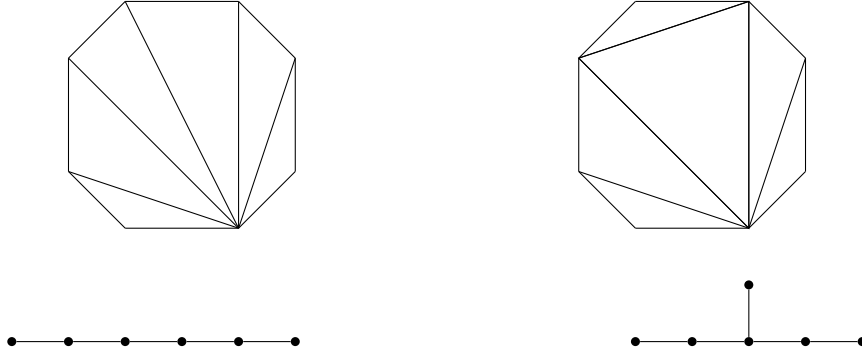
1.3.3 Triangulations

The next class of graphs we will examine are the triangulations of regular n -gons. We will see that their tree number has an interesting connection to the Fibonacci sequence, given by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$. A direct consequence of this definition is the relation

$$F_{n+2} + F_{n-2} = 3F_n. \tag{5}$$

We define a *triangulation* of a regular n -gon as a partition of the figure into non-overlapping triangles using diagonals. Any triangulation of a regular n -gon has $n - 2$ triangles and a triangle is said to be *interior* if none of its sides lie on the original n -gon.

Example 10. *Two triangulations of an octagon, along with their weak duals:*



This example shows two different triangulations of an octagon, one with no interior triangles and the other with exactly one interior triangle. The weak dual is useful here so that we can easily keep track of the configuration of triangles.

In this section, let $G = G_n$ denote a triangulation of a regular n -gon. We explore the number of spanning trees of such graphs using the Cycle Intersection matrix. We denote by T_n the tridiagonal matrix with -1 on the subdiagonal and superdiagonal, and 3 on the main diagonal. For a triangulation with no interior triangles, the weak dual graph is the path on $n - 2$ vertices, so $C(G) = T_{n-2}$. This matrix will also appear as blocks in $C(G)$ for triangulations with interior triangles. The fan on n vertices is a special case of a triangulation with no interior triangles [4], and it has been shown that $\det T_n = F_{2(n+1)}$, [11]. Since T_n is tridiagonal, its determinant satisfies a recurrence relation, as seen in the next lemma. Recall that $|T_n|$ denotes the determinant of the matrix T_n .

Lemma 9. $|T_n| = 3|T_{n-1}| - |T_{n-2}|$.

Proof. We compute the determinant across the first row:

$$|T_n| = \begin{vmatrix} 3 & -1 & 0 & 0 & \dots \\ -1 & 3 & -1 & 0 & \dots \\ 0 & -1 & 3 & -1 & \dots \\ \vdots & \vdots & \vdots & & \ddots \end{vmatrix}$$

$$\begin{aligned}
&= 3|T_{n-1}| + \left| \begin{array}{ccccc} -1 & -1 & 0 & \dots & 0 \\ 0 & & & & \\ \vdots & & \mathbf{T}_{n-2} & & \\ 0 & & & & \end{array} \right| \\
&= 3|T_{n-1}| - |T_{n-2}|
\end{aligned}$$

□

This recurrence relation along with Equation 5 yields the following result.

Theorem 10. *Let $G = G_n$ denote a triangulation of a regular n -gon with no interior triangles. Then, $\kappa(G) = F_{2(n-1)}$.*

Proof. We will proceed by induction on n . G_3 is a triangle, hence $C(G_3) = (3)$ and since $F_4 = 3$, the result holds. Now,

$$\begin{aligned}
|T_n| &= 3|T_{n-1}| - |T_{n-2}| \\
&= 3F_{2n-4} - F_{2n-6} \\
&= F_{2n-2}.
\end{aligned}$$

□

In the next two sections, we will see that the Cycle Intersection matrix is not quite tridiagonal for triangulations with interior triangles. However, the structure of the matrix lends itself nicely to the use of Laplace's expansion theorem. This theorem allows us to calculate the determinant using multiple rows and columns (see [12] for details).

Theorem 11. *Let A be an $n \times n$ matrix. Let $A(i_1 i_2 \dots i_k | j_1 j_2 \dots j_k)$ denote the $k \times k$ submatrix of A consisting of the intersection of rows i_1, i_2, \dots, i_k and columns j_1, j_2, \dots, j_k . Let $M(i_1 i_2 \dots i_k | j_1 j_2 \dots j_k)$ denote the submatrix which is complement*

to $A(i_1 i_2 \dots i_k | j_1 j_2 \dots j_k)$. That is, $M(i_1 i_2 \dots i_k | j_1 j_2 \dots j_k)$ is formed by deleting rows i_1, i_2, \dots, i_k and columns j_1, j_2, \dots, j_k from A . Define

$$\overline{A} = (-1)^{i_1+i_2+\dots+i_k+j_1+j_2+\dots+j_k} M(i_1 i_2 \dots i_k | j_1 j_2 \dots j_k).$$

Then, for every fixed set of row indices $1 \leq i_1 < i_2 < \dots < i_k \leq n$,

$$\det(A) = \sum_{1 \leq j_1 < j_2 < \dots < j_k \leq n} \det A(i_1 i_2 \dots i_k | j_1 j_2 \dots j_k) \det \overline{A}(i_1 i_2 \dots i_k | j_1 j_2 \dots j_k)$$

Essentially, this expansion is performed by first choosing a fixed number $k \in [n]$. Then we find a set of k columns from the first k rows and its complement in the remaining $n - k$ rows, and multiply the determinants of these submatrices. We do this for every set of k columns, and sum as in the theorem. This expansion will be particularly useful in calculating the determinant of $C(G)$ because of the structure of the matrix. An example of its use will follow Theorem 14.

1.3.4 Triangulations: One Interior Triangle

We now consider triangulations of regular n -gons with exactly one interior triangle. Note that such graphs are only possible with $n \geq 6$ and that the number of distinct triangulations of this form is the number of 3 partitions of $n - 3$. In 1981, an equivalent result to Lemma 13 below was shown using a reduction formula that makes use of the deletion-contraction formula [4]. However, this method results in computing spanning trees of multiple graphs, some of which contain multiple edges. The results here depend only on the configuration of the graph and a function, $T(n)$, defined below.

Suppose that for such a triangulation, there are u, v , and w triangles on the three sides of the interior triangle, respectively. Consider the general shape of the triangulated graph G as in Figure 2.

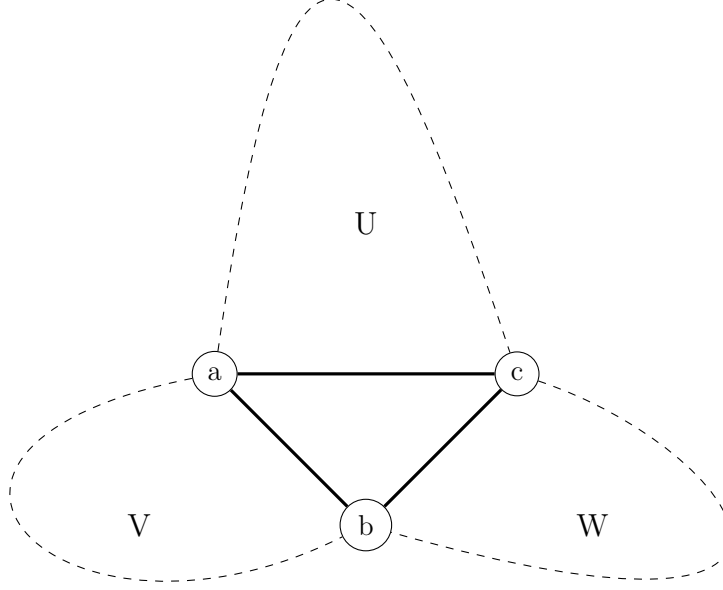


Figure 2. Layout of G

Label the vertices of the interior triangle a, b , and c . Let U, V and W be the subgraphs of G that lie on either side of the interior triangle, so that $ab \in V$, $bc \in W$, and $ac \in U$. In order to impose an advantageous structure on the matrix, we label the triangles starting with the top triangle in U , labeling down toward the interior triangle (but not labeling the interior triangle), then labeling the remaining triangles in order from V through the interior triangle to W . With this labeling, $C(G)$ has the structure:

$$C(G) = \left(\begin{array}{c|c} T_u & -1 \\ \hline -1 & T_{v+w+1} \end{array} \right)$$

We define $T(n) = F_{2(n+1)}$ so that $T(0) = 1, T(1) = 3, T(2) = 8$, and so

on. Notice that this function simply keeps track of the even indexed Fibonacci numbers. We will use this function in a symmetric expression for the number of spanning trees in a triangulation with one interior triangle. Here, we will make use of Laplace's expansion theorem.

Lemma 12. *For integers m and n , $T(m+n) = T(m)T(n) - T(m-1)T(n-1)$.*

Proof. Regard $T(n)$ as the determinant of the tridiagonal matrix T_n . We compute $T(m+n)$ using Laplace's expansion theorem. Let $T = T_{m+n}$ be the $(m+n) \times (m+n)$ tridiagonal matrix. Suppose that $m \geq 2$ and let $k = m$. That is, when computing the determinant, we will choose a set of m columns from the first m rows of T . For $j > m+1$, column c_j in the first m rows is a zero column. Similarly, for $j < m$, column c_j in the bottom n rows is a zero column. Thus, to contribute a nonzero term to the determinant, we must choose the first $m-1$ columns from the first m rows and then we may choose either column m or column $m+1$ from these rows. Counting only the submatrices with nonzero determinant, we have

$$\begin{aligned} \det(T) &= \det T(1, 2, \dots, m | 1, 2, \dots, m) \overline{T}(1, 2, \dots, m | 1, 2, \dots, m) \\ &+ \det T(1, 2, \dots, m | 1, 2, \dots, m-1, m+1) \overline{T}(1, 2, \dots, m | 1, 2, \dots, m-1, m+1) \end{aligned}$$

In terms of the matrix, this gives us

$$|T_m||T_n| - \left| \begin{array}{c|c} \boxed{\mathbf{T}_{m-1}} & \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix} \\ \hline \begin{matrix} 0 & 0 & \dots & -1 & -1 \end{matrix} & \begin{matrix} -1 & -1 & 0 & \dots & 0 \\ 0 & \boxed{\mathbf{T}_{n-1}} & & & \end{matrix} \end{array} \right|$$

Note that the sign of $\overline{T}(1, 2, \dots, m | 1, 2, \dots, m)$ is $(-1)^{2(1+2+\dots+m)}$, which is always positive, and the sign on $\overline{T}(1, 2, \dots, m | 1, 2, \dots, m-1, m+1)$ is $(-1)^{2(1+2+\dots+m)+1}$,

which is negative. Further, computing the determinants of the last two matrices by expanding down the columns with only one -1, we have $T(m-1)T(n-1)$. Thus, $T(m+n) = \det(T) = T(m)T(n) - T(m-1)T(n-1)$.

□

Using this function, we have a formula for the number of spanning trees in a triangulation with exactly one interior triangle.

Lemma 13. $\kappa(G) = T(u)T(v+w+1) - T(u-1)T(v)T(w)$

Proof. The proof of this lemma is essentially the same as the proof of Lemma 12 above, but with $k = u$ in Laplace's expansion theorem. □

The main theorem below gives a symmetric expression for the number of spanning trees in a triangulation with one interior triangle. Note that this expression is equivalent to Lemma 13.

Theorem 14. $\kappa(G) = T(1)T(u)T(v)T(w) - T(u-1)T(v)T(w) - T(u)T(v-1)T(w) - T(u)T(v)T(w-1)$

We provide two proofs of this theorem, an algebraic proof using the lemmas above and a bijective proof.

Proof. Using Lemma 13 and repeated use of the identity in Lemma 12, we have

$$\begin{aligned}
\kappa(G) &= T(u)T(v+w+1) - T(u-1)T(v)T(w) \\
&= T(u)(T(w)T(v+1) - T(w-1)T(v)) - T(u-1)T(v)T(w) \\
&= T(u)T(w)(T(v)T(1) - T(v-1)T(0)) - T(w-1)T(u)T(v) - T(u-1)T(v)T(w) \\
&= T(1)T(u)T(v)T(w) - T(v-1)T(u)T(w) - T(v)T(w-1)T(u) - T(u-1)T(v)T(w)
\end{aligned}$$

□

We now prove Theorem 14 bijectively.

Proof. (Bijective Proof) We will show that $\kappa(G) + T(v-1)T(w)T(u) + T(v)T(w-1)T(u) + T(v)T(w)T(u-1) = 3T(v)T(w)T(u)$. Since U, V , and W each have no interior triangles, $T(u)$, $T(v)$, and $T(w)$ are the number of spanning trees in U, V , and W , respectively. Now, we create a set of three trees by choosing one from each of U, V , and W . For each set of this type, choose one tree to be special by marking the unique path between the two vertices from the interior triangle (either the path from a to b , b to c , or a to c). Let $X = \{(p, B_1, B_2, B_3)\}$ where B_1, B_2, B_3 are spanning trees in U, V and W , respectively, and p is the unique path in B_1 or B_2 or B_3 from a to b , b to c , or a to c , respectively. Hence, $3T(v)T(w)T(u) = |X|$.

Let \mathcal{T} be a spanning tree in G . In \mathcal{T} , there must be exactly one pair from $\{a, b, c\}$ which does not belong to a path inside that pairs subgraph (U, V , or W), otherwise \mathcal{T} would include a cycle. Without loss of generality, suppose that this pair is b, c . For our first bijection, we map \mathcal{T} to $\mathcal{T} + bc$. Note that $\mathcal{T} + bc \in X$ and is one of the elements with marked path of length 1. Thus, the spanning trees of G are in bijective correspondence with the trees in X with marked path length 1.

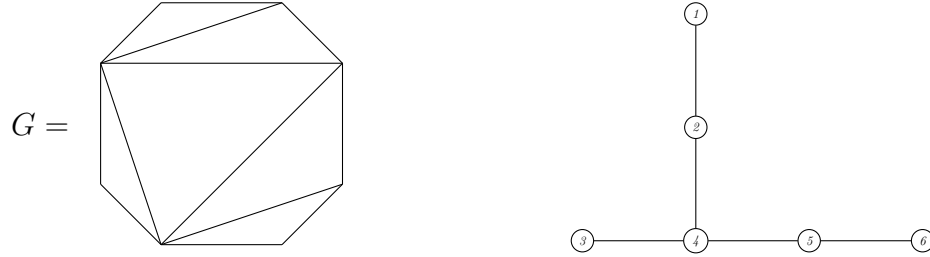
Next we will consider the subgraph U . Denote the vertex different from a but adjacent to c in U by d . If c has other neighbors in U besides a and d , then switch the labels on a and c . Let G' be the graph obtained from G by removing edge cd . Now, G' contains subgraphs U', V, W , where U' has $u-1$ triangles and no interior triangles. Create a spanning tree, \mathcal{T}' from G' by combining spanning trees from U', V , and W . By construction, \mathcal{T}' contains no path in U' from a to c . Now, let M be the path-marked tree created from $\mathcal{T}' + cd$ by marking the unique path from a to c . Note that this path has length greater than one. Hence, the trees counted by $T(u-1)T(v)T(w)$ are in bijective correspondence with the trees in X with marked path from a to c of length greater than one.

The remaining bijections are constructed in a similar fashion for the terms $T(v-1)T(w)T(u)$ and $T(v)T(w-1)T(u)$.

□

As an example, we will compute $\kappa(G)$ where G is a triangulation of an octagon with one interior triangle with $v = 1$ and $w = u = 2$.

Example 11. *Below we have the triangulation along with its weak dual graph.*



$$C(G) = \begin{pmatrix} 3 & -1 & 0 & 0 & 0 & 0 \\ -1 & 3 & 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & -1 & 0 & 0 \\ 0 & -1 & -1 & 3 & -1 & 0 \\ 0 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & -1 & 3 \end{pmatrix}$$

Notice that $C(G)$ is almost a tridiagonal matrix. We will use Laplace's expansion on the first two columns, that is, we choose 2 columns from the first 2 rows. The sparsity of the matrix forces us to choose column 1 from the first two rows and then we may choose column 2 or column 4. All other choices result in a zero determinant, so the expansion is as follows:

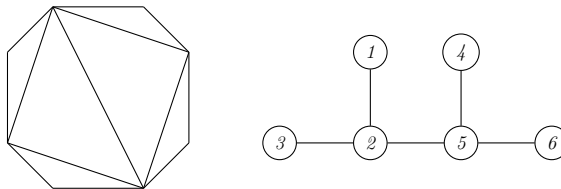
$$\begin{aligned}
\det C(G) &= |T_2| |T_4| - \begin{vmatrix} 3 & 0 \\ -1 & -1 \end{vmatrix} \begin{vmatrix} 0 & 3 & 0 & 0 \\ -1 & -1 & -1 & 0 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & -1 & 3 \end{vmatrix} \\
&= 8(55) - 3(3)(8) \\
&= F_6 F_{10} - F_4 F_4 F_6 \\
&= 378
\end{aligned}$$

We emphasize that the elegance of this result lies not in the final number but in the representation of that number by the Fibonacci sequence, which appears with the use of Laplace's expansion theorem.

1.3.5 Triangulations: Two Interior Triangles

The case where a triangulation contains two interior triangles is a bit more complicated than with one interior triangle, but the above method works fairly well. The key to using the method above is to have a labeling of the triangles (i.e. a labeling of the vertices of the weak dual) that gives rise to a matrix with a desirable structure for using Laplace's expansion theorem. We focus here on the case where the two interior triangles are adjacent, that is, they share an edge. In this case, the weak dual can be thought of as two path graphs connected by a new edge. The most effective way to label the vertices of the weak dual is to label each path in sequential order, as shown in Example 12.

Example 12. *An octagon with two interior triangles and the labeling of its weak dual:*



The next figure shows the general layout of G and the labeling of its weak dual. Suppose that there are u, v, w and x triangles on the sides of the two interior triangles and let U, V, W and X be the subgraphs of triangles on these sides. We label one side of the weak dual at a time, just as we did in Example 12.

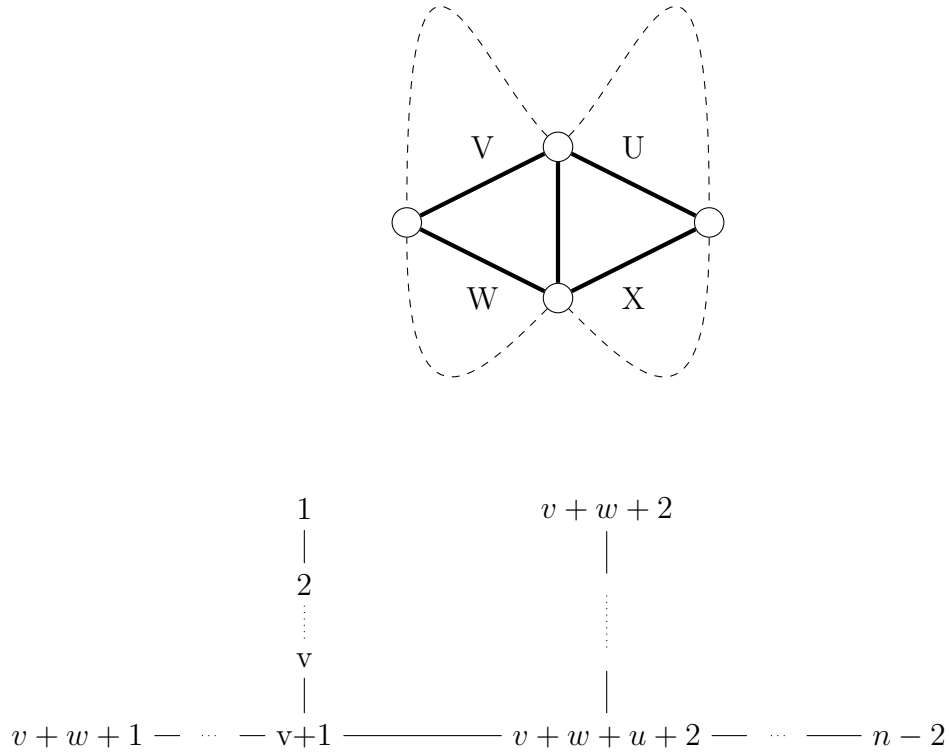


Figure 3. Labeling in the case of two adjacent interior triangles.

Theorem 15. *For a triangulation of a regular n -gon with two adjacent interior triangles as in Figure 3, $\kappa(G) = T(v+w+1)T(u+x+1) - T(w)T(x)T(u)T(v)$.*

Proof. This proof is essentially the same as the proof of Lemma 12, but with $k = v + w + 1$ in Laplace's expansion theorem. \square

1.4 Conclusions

Although there are many tools for calculating the number of spanning trees in a graph, the method we have described here has the advantage of being more straightforward and elegant. This method extends the collection of tools available for computing the tree number, and as we saw above, reveals particularly interesting results for certain classes of graphs.

Since the elegance of the results on the triangulations lies not in the final number but in the decomposition of that number, methods for continuing this work should be chosen with this idea in mind. We believe that the method used in Sections 1.3.4 and 1.3.5 is not the most efficient for triangulations with more than two adjacent interior triangles, since Laplace's expansion theorem is effective given a certain labeling of the triangles. However, methods such as those used by Modabish and Marraki in [13] which compute the tree number by first breaking the graph into smaller pieces, may be useful here. It may be practical to break up these triangulations into smaller triangulated n -gons with fewer interior triangles and use the results above along with the results in [13]. Further, there are many recurrence relations on the Fibonacci numbers which may also be useful for triangulations.

There are other classes of graphs which may yield interesting results using the methods here. For instance, we may consider "grid" graphs which are made up of triangles or pentagons rather than squares. These other grid graphs may reveal interesting numbers when the above methods are applied.

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CHAPTER 2

Postsecondary Mathematics Education

2.1 Introduction

In this chapter, we address our work in postsecondary mathematics education. In the Fall of 2012, we began a remediation and assessment program for our calculus course in order to address the problem of underprepared students.

2.2 Problem: Underprepared Students

Many high school students in the United States are graduating with deficiencies in mathematics and are unable to successfully complete a first year calculus course [1]. An inadequate background in mathematics puts students in a position where they need remediation in order to succeed in STEM (science, technology, engineering and mathematics) disciplines. As a result, these students are more likely to leave a STEM major and less likely to complete any degree than those who have the necessary background [2]. Research highlights that a strong STEM-educated workforce is not only vital to the nation's economic integrity, but also to the nations security [3], raising concern over high rates of attrition in STEM disciplines. Since calculus is considered a gateway course to the sciences, it is crucial that students have the proper background to succeed, and that those who do succeed are able to retain and apply their knowledge in subsequent courses.

A first year calculus course is known to be one of the most challenging courses for college students. Approximately 300,000 college and university students in the United States take an introductory calculus course each fall, and of those, 28% earn a grade of D, F, or W [4]. These high failure rates have led to a national concern not only over inadequately prepared students but also over the declining numbers of students studying a STEM discipline [5].

Many students enter calculus with weak algebra skills and while some fail, others may finish the course without ever strengthening those skills. As a result, these students are more likely to leave a STEM major than those who have the necessary background [2]. Research has indicated that mathematics plays a prominent role in the attainment of a college degree and that success in mathematics is a significant factor in career opportunities [1]. With such a large number of students taking calculus as a foundation for their engineering, physics, chemistry, and mathematics majors, it is necessary to address this issue.

At the University of Rhode Island, approximately 500 to 900 students enroll in MTH 141 (Calculus I for STEM majors) each year. Many of these students continue on to more advanced mathematics courses in preparation for their major courses. Since 2008, the unproductive rate for MTH 141 (grades of D+, D, F, I, or NW) has ranged from 31% to 50%. At the same time, the unproductive rate of students coming out of the precalculus course has ranged from 17% to 39%. This data suggests that many students are entering calculus underprepared, and as a result, are not succeeding in the course.

Despite the national concern over underprepared students, little research has been published on what we can do to help the students who enter mathematics courses without the prerequisite knowledge. The research that does exist focuses primarily on the fact that students graduate high school unprepared for college level mathematics. There is also an emphasis on the importance of placing students in the proper courses [6], however, proper placement cannot always be enforced. Although URI does have placement exams, they are administered online and unproctored, and the placement is not strictly enforced. Some students place into courses for which they are not prepared, and others choose to enroll in a higher level course than their exam would indicate they are ready for. As a result, many

students are unsuccessful in a course and must repeat it.

Some research suggests that remedial courses, workshops, and extra class time are partial solutions to this problem, but these methods can be expensive [7]. It is important to remediate underprepared students in a feasible way which does not disrupt the flow or content of the course, and such remediation must be done in a way that helps weaker students while keeping strong students interested and challenged [7]. Therefore, this research will address the question: How can we provide remedial instruction during a calculus course for students who are underprepared for calculus?

2.3 Literature

The literature contains many definitions of post-secondary mathematical readiness. Some organizations value high school graduation and GPA, while others emphasize standardized test scores and college placement exam performance [8]. Conley [9] points out that student attitudes, study skills, and self-awareness also need to be considered when defining college readiness. In an effort to be more comprehensive, Conley [9] defines college readiness as “The level of preparation a student needs to enroll and succeed, without remediation, in a credit-bearing general education course at a postsecondary institution that offers a baccalaureate degree” (p. 5). For the purpose of this work, the term mathematical readiness will mean an individual’s ability to succeed in college level mathematics without remedial coursework.

In order for students to obtain the background knowledge necessary to succeed in STEM disciplines, they must first be placed in the proper courses [10]. Students enter universities with varying mathematical backgrounds. Even though many have taken a precalculus course, the material they learned differs based on the institution, instructor, and the students knowledge and maturity [6]. Conse-

quently, proper placement must be based on more than the courses a student has taken. Many universities use SAT, ACT, or AP scores as a placement tool for first semester mathematics courses [10]. However, such measures of knowledge may not be current, as a student may continue to take mathematics courses after taking these exams, or may have taken the exams in their junior year of high school with no mathematics courses in their senior year [10]. In an effort to place students properly, many universities have internal placement exams administered by their departments.

For universities where placement is not enforceable, a just-in-time approach to remediation is necessary to help those students who enroll in a course for which they are underprepared. Postsecondary remediation is a controversial topic for many reasons. Although it fills the gap in knowledge, some argue that it wastes tax dollars, lowers academic standards, and devalues the credentials of faculty [11]. Further, it has been shown that remedial courses simply do not work. More than 50% of students entering two-year colleges and 20% entering four year colleges are placed in remedial courses, while fewer than 1 in 10 graduate from community colleges within three years and about a third complete a bachelor's degree in six years [12]. One nonprofit organization, Complete College America, has worked with states to improve student success and has done extensive work regarding remediation. This organization points out that remedial courses may be to blame for unchanged college completion rates even though enrollment has increased. As a method of reform, they have proposed enrolling more students in college-level courses with just-in-time support rather than sending students to remedial courses [12]. This approach is particularly useful for universities, who typically offer few, if any, remedial math courses for credit.

Remediation during a course can take on many forms. The mathematics

department at California State University, Los Angeles, used workshops as an intervention tool to improve calculus success rates [13]. This department experienced high failure rates in calculus courses which they attributed to students' inexperience with problem solving, and lack of class time for instructors to go over the solutions to practice problems. As a first step, the faculty decided to change from 100 minute lectures two days per week to 50 minute lectures four days per week. Although the students seemed more motivated with the new schedule, they still lacked the problem solving skills necessary to succeed [13].

In order to address the issue of high failure rates more fully, the faculty implemented required workshops for calculus students. The workshops met twice a week, were run by an experienced teaching assistant, and were designed as a way for students to actively gain problem solving experience [13]. The data collected indicated that the workshops were successful in lowering the failure rate, and in particular, indicated that those students with a grade of C in the previous course were able to pass the next course [13]. The Rochester Institute of Technology adopted a similar workshop model which increased student success rates by over 16% [7].

While workshops and daily class meetings may be a successful way to remediate, such methods may be expensive, impractical, or cause scheduling conflicts [7]. In order to address the issue of underprepared students in their calculus sequence, Clarkson University has implemented a gateway exam to ensure that students master the necessary background material [14]. This gateway testing program, called the Calculus Absolutely Basic Competencies (ABCs), consists of not only a gateway exam but also resources for students who need extra practice. In order for students to pass calculus with a C or better at Clarkson University, they must score at least a 90% on the ABCs (with no partial credit; answers are fully right or

wrong). That is, students who pass the ABCs may earn any grade in the course, but those who do not pass are unable to earn any grade higher than a D+ [14]. This method guarantees that students who earn a C or better in calculus are truly prepared to continue in mathematics. The initial exam helped the students identify weaknesses in background, and then the students were allowed to take the exam as many times as necessary during the semester. To guide the students who did not pass, the department offered many resources. Practice exams and solutions were available to the students, and optional ABCs review sessions were given twice each semester. Further, in order to help the students see the importance of background material, ABCs topics were identified during the calculus lectures.

During the first two years of this program, about 60% of the students passed the exam within three attempts, and about 80% passed within five attempts [14]. Since few students who took the exam more than five times passed, the exam was only given six times during the third year, which reduced administrative work and allowed instructors to focus fully on the calculus material after a few weeks [14]. Clarkson's program has been expanded to their Calculus I and Calculus II courses, and improvements have been made such as computer generated exam questions and the addition of a companion course meeting two hours per week to help the students who have difficulty passing.

To address the gap in literature, we have adopted a model similar to the Calculus ABCs model used at Clarkson University (which is also similar to a model used at West Point) and study the effect of a competency based exam on student performance.

2.4 The Precalculus Competency Exam

The Precalculus Competency Exam (PCE) was designed as a remediation tool for Calculus 1 (MTH 141). The purpose of the exam is to identify individual

students weaknesses and provide them with a way to self-remediate and fill in the gaps in topic areas that are necessary for success in calculus. We have run this program every semester since Fall 2012, but the work here is focused on the Spring 2013 semester.

2.4.1 The PCE

The PCE is made up of ten content areas, or competencies, which are necessary for success in calculus. These competencies are aligned with the content of the course, and are outlined in Table 1.

Table 1. PCE Topics

PCE Competency	Corresponding Topic In MTH 141
1 - Functions	Functions, difference quotient
2 - Factoring and Expanding	Limits
3 - Graphing and Quadratics	Limits, continuity,integral as area, tangent lines
4 - Radicals and Exponents	Simplifying expressions to differentiate
5 - Straight Lines	Tangent line approximation
6 - Logarithms	Evaluating derivatives
7 - Algebra	Solving in implicit differentiation
8 - Inequalities	Derivative tests
9 - Trigonometric Functions	Parametric equations, related rates
10 - Real Numbers	Optimization and modeling, related rates

The original exam (see Appendix A) consists of two questions for each of the ten competencies, for a total of 20 questions. The questions are open-ended and graded as correct or incorrect (no partial credit). The students take the original exam during the second week of class and must score at least 80% in order to pass. When the exams were graded, the instructor identified the weak areas for each student. A student who did not pass the original exam was required to pass a mini test (consisting of two questions, see Appendix B) in each competency that was

not mastered on the original exam. Opportunities to take these mini tests were scheduled twice each week. Each student was able to take two different mini tests per week, for the remainder of the semester, if necessary, to pass all competencies. Students lost points for each competency they had not passed by the end of the semester. By breaking up the exam into the mini-tests, students were able focus on a particular area each week, rather than having to take the entire exam again. In order to support the students who did not pass the original exam, we offered a special supplemental instruction session each week. Further, the tutoring center was equipped with old PCE's for the students to work on with tutors. Instructors were also encouraged to point out why the PCE topics are relevant during their calculus lectures.

We started this program in the Fall of 2012, and the data presented here is from Spring 2013. Our research and findings have shaped the current program and its policies.

2.4.2 Research Questions

The following questions are typically in the forefront of scholars minds in postsecondary mathematics education, and thus, our initial data collection was motivated by the following:

1. Does student proficiency on PCE improve over time?
2. Among the students who did not pass the original PCE, how does proficiency on mini tests affect performance on related calculus exam questions? Does it matter when they demonstrated proficiency?
3. Does the PCE help solidify a student's foundation and help them as they move through the calculus sequence and into their STEM major?
4. How can we modify the program to better help the students?

5. Do the students retain the information learned on the PCE's?
6. Are students in a particular major or living learning community performing better than other students?
7. Is there a correlation between PCE performance and a student's final grade?

In this thesis, we address research questions 1,2,4, and 5.

2.4.3 Data Collection

Student progress was tracked each week in a dynamic spreadsheet which was used to record which exam each student took in a given week and whether or not the student passed that exam. Exams were handed back to students at the end of each week and their scores were also recorded on our online course management system.

In addition to the weekly PCE data that was collected, we also re-tested the ten competencies on the final exam. Since the final exam was multiple choice, this data was easy to gather. This data, which will be discussed later, gave us an indication of whether or not students retained the material. Further, each of the three mid term exams in the course contained calculus questions which required knowledge of some PCE topics. For each of the midterm exams, we recorded whether or not each student was able to complete the underlying task of the question. The example below shows an exam question along with the PCE competency that corresponds to that topic.

Example 13. *Exam 2 asked students to find $\frac{dy}{dx}$ for $5y - \tan(y) + 3x = 10x^3y^2 + 14$. PCE competency 7 is Algebra. This competency asked students to solve an equation for a specific variable, if possible, or explain why it is not possible.*

For this example, we recorded if the student was able to solve for $\frac{dy}{dx}$ correctly, even if they made a differentiation error.

2.4.4 Data Analysis

The analysis here is based on data from the spring semester 2013, in which there were 286 students enrolled in the course. The first thing we found when we examined the data was that, by giving the mini tests each week, the students were able to progress through the competencies fairly well. Table 2 below shows the number of students who passed each competency on the original exam.

Table 2. Results of Original Exam

PCE Exam	# of students who passed	% of total students
1 - Functions	89	31
2 - Factoring	194	68
3 - Graphing	77	27
4 - Exponents	55	19
5 - Lines	112	39
6 - Logarithms	134	47
7 - Algebra	81	28
8 - Inequalities	43	15
9 - Trigonometry	81	28
10 - Real Numbers	148	52

The results of the original exam gave us a sense of what the students knew upon entering the course. One particular reason for concern is that only 31% of the students mastered the competency on functions, so we may need to address this issue along with some other competencies at the precalculus level. In comparison, the next table shows the number of students who mastered each competency by the end of the semester. We can see that by giving the students multiple opportunities to take the exams each week, many of them were able to improve. We must keep in

mind, however, that some students may have dropped the course before completing all PCE's.

Table 3. Results by End of Semester

PCE Exam	# of students who passed	% of total students
1 - Functions	196	69
2 - Factoring	256	90
3 - Graphing	210	73
4 - Exponents	165	58
5 - Lines	206	72
6 - Logarithms	196	69
7 - Algebra	177	62
8 - Inequalities	133	47
9 - Trigonometry	161	56
10 - Real Numbers	216	76

In order to address research question 2, we examined PCE data and related calculus exam questions. For example, on Exam 1, the students were asked to calculate $\lim_{x \rightarrow -2} \frac{x^2 - 2x - 8}{x^2 - 4}$. Since the students had not yet learned L'Hôpital's rule, they must first factor the numerator and denominator and cancel a common factor. When we graded the exam, we recorded if each student factored correctly and then we were able to compare this data with whether or not each student had passed PCE 2 by that time. What we found is that there was a very small percentage (about 4%) of students who passed PCE 2 but were not able to correctly answer this question (this is the PO category in Figure 4 below). Hence, we saw that students who passed this competency by the time of the exam were likely able to apply their knowledge to the related exam question. The results of this particular comparison are seen in the Figure 4.

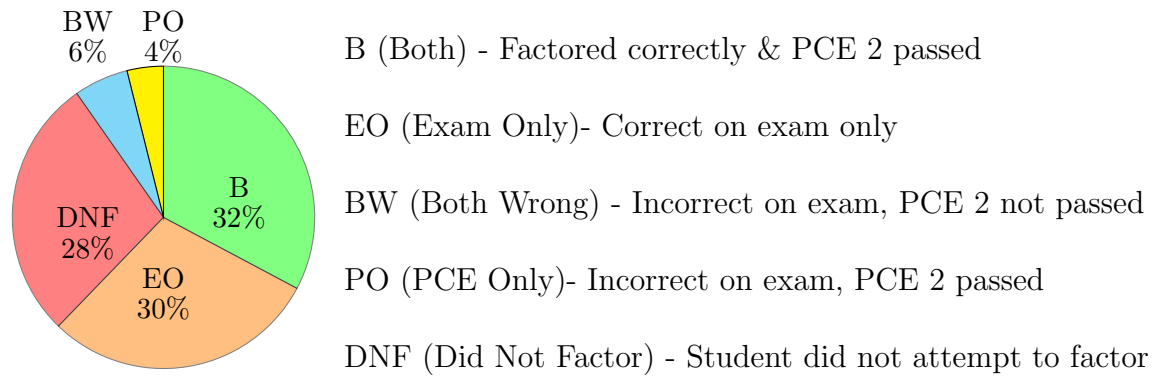


Figure 4. Exam 1 and PCE 2

We were pleased with the percentages in the B category and the PO category, which indicate success of our program. It is important to keep in mind that some of the students in the EO category may not have had a chance to try PCE 2 before the exam. In the future, imposing deadlines for students to pass certain competencies may give us a better sense of this data.

To address research question 5 about whether the students are retaining the information learned on PCE's, we re-tested the ten competencies on the final exam. The final exam was multiple choice and the first ten questions were PCE questions, one from each competency. In the analysis above, we were concerned with how many students were successful, but here it is more meaningful to look at how many students did not retain the material. We asked ourselves the question, "Of the students who had passed the competency by the end of the semester, how many of them answered the corresponding final exam question *incorrectly*?" These numbers appear in Table 4 and the corresponding final exam questions can be found in Appendix D. What we found is that most students retained the material, although there are a few numbers here that are higher than we would like.

Table 4. Results of Final Exam

PCE competency	# who answered incorrectly	% of students taking final
1 - Functions	68	28
2 - Factoring	25	10
3 - Graphing	10	4
4 - Exponents	17	7
5 - Lines	50	20
6 - Logarithms	48	20
7 - Algebra	42	17
8 - Inequalities	40	16
9 - Trigonometry	19	8
10 - Real Numbers	9	4

Again, the first competency is concerning to us, since 28% of the students did not retain that material. Because of this high percentage on such a vital topic, we may need to consider having this topic appear on multiple competencies in the future, so that students are tested on it throughout the semester. It may also be interesting to track student progress on these topics through subsequent calculus courses to see if material is retained for a longer period.

2.5 Program Analysis Using Graph Theory

As with any program, it is important for us to reflect on our findings and modify the PCE's as appropriate. One question that comes up in this type of remediation is if we are asking the students to learn two courses worth of material at once. To avoid overwhelming the students with too many exams or too much remediation, we can reduce the number of competencies. Further, the program is costly to run as it is. Each week, we copy, grade, enter scores, and hand back hundreds of exams which consumes resources and time. In an effort to cut down on the number of exams students need to take without compromising the original topics, we examined the ten competencies using graph theory.

As a measure of difficulty for each competency, we use the total number of students who passed that competency on the original exam. These numbers were shown in Table 2. For our purposes, we regard the competencies with more students passing as the easier competencies. For example, we can see that competencies 2 and 10 were the easiest for students to pass in this particular semester. Note that the difficulty of exams by this measure is likely to change given a different group of students.

Using a graph model, we are able to visualize the relationships among the ten competencies with regard to student performance. The graph below consists of ten vertices, one for each competency. An edge exists between two competencies if at least 20% of the students passed both on the original exam. The edges are weighted in the following manner: if 20% – 29% of the students passed both, then the edge weight is 1, if 30% – 34% passed both, then the edge weight is 2, and if at least 35% passed both, then the edge weight is 3. With the help of the network software ORA, we were able to visualize the graph from the spreadsheet data.

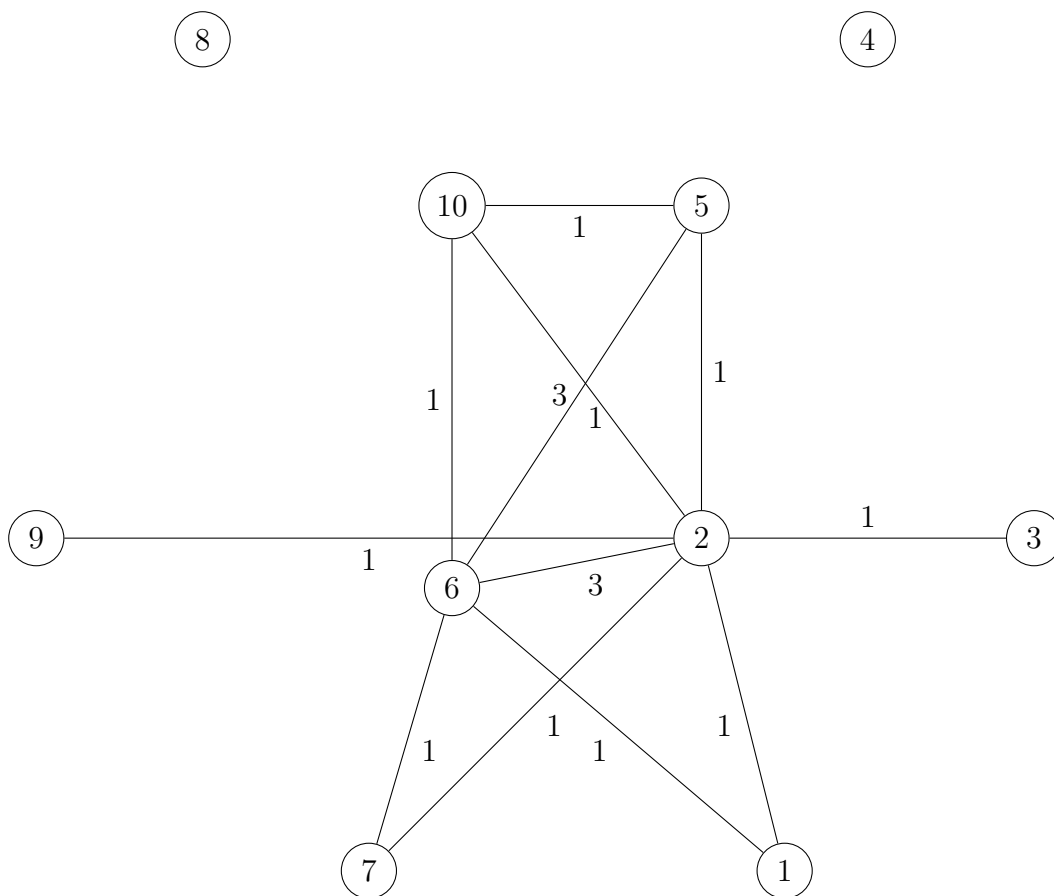


Figure 5. PCE Graph

In order to reduce the number of mini tests a student needs to complete, we would like to combine some competencies into one exam, while keeping the difficulty of the exams relatively even. To make the decision of which exams to combine, we examined a special substructure of a graph called a clique.

Definition 4. *A clique is a complete subgraph. That is, a clique is a subgraph where every pair of vertices is adjacent.*

Using ORA, we were able to find all of the cliques in the PCE graph. It is fairly easy to see that there are three cliques of size at least three: $\{1, 2, 6\}$, $\{2, 6, 7\}$, and

$\{2, 5, 6, 10\}$. This last clique, which we will call C , is particularly revealing. Given that these four vertices form a clique and that two of the edge weights are three, these strong ties between the four vertices in C suggest that these competencies can be combined into one. We will call the mini test consisting of these four competencies Exam C . From there, we can cut down the number of questions on this larger exam as appropriate. In order to measure the difficulty of this new Exam C , we counted how many students passed these competencies, as shown in the table below.

Table 5. Difficulty of Exam C

Number of Competencies in $\{2, 5, 6, 10\}$ Passed	# of Students
Passed Zero	34
Passed Exactly One	60
Passed Exactly Two	77
Passed Exactly Three	74

We can see from Table 5 that there are 94 students who mastered fewer than two of these competencies, and therefor really struggled with the topics in PCE's 2, 5, 6, and 10. Since these four topics are easier than the others, Exam C would be a good place for these 94 students to start when taking mini tests. We can compare the numbers above to the difficulty of the other six exams (as shown in Table 2) and see that there is a relatively even measure of difficulty. A proposed version of Exam C can be found in Appendix C .

The question arises of why a graph is useful here. Although we can see from the spreadsheet that topics 2, 5, 6, and 10 were the highest-ranking in terms of how many students passed, we do not see the strong connection among all four exams without the graph. Further, the five highest ranking exams do not make

up a clique, reaffirming that one could not simply look at Table 2 to reveal these substructures. Hence, running ORA to find the cliques has given us a way to visualize the relationships among the topics in a new way, which has allowed us to make a decision regarding the structure of the program.

We also created graphs using students as vertices and edge weights as the number of common exams two students had passed. These graphs, however were not particularly revealing. The typical measures of centrality (betweenness, degree, closeness) and even the cliques only picked up the top students in the class, which we could easily see using a spreadsheet or a bar graph. Another interesting idea we considered was whether or not there was any correlation between a student's major and PCE performance. The data we were able to gather, however, had many majors not listed. Perhaps a student survey at the beginning of the semester would give us a more accurate idea of each student's major at the time of the course.

2.6 Conclusions

In this work, we described the use of and analyzed the results of a remediation program for calculus. Here we provide a summary of our findings.

- Does student proficiency on PCE improve over time?

In general, we found that students took advantage of the opportunity to review topics and take mini tests to improve their understanding.

- Among the students who did not pass the original PCE, how does proficiency on mini tests affect related calculus exam questions? Does it matter when they demonstrated proficiency?

Here we examined one PCE topic and its related exam question and found that only 4% of students who took the exam has previously passed PCE 2 but were unable to apply that knowledge to the calculus question.

- How can we modify the program to better help the students?

We used a graph to model the PCE program and proposed a way to combine certain PCE's in order to cut down on the number of exams a student needs to take.

- Do the students retain the information learned on the PCE's?

By testing the PCE topics on the final exam, we were able to see exactly which topics were retained and which were not. Some further program changes could include testing certain topics in multiple ways to raise retention rates.

2.7 Future Work

In terms of continuing the PCE program at the University of Rhode Island, there are a few things we would like to do in the future. First, we need to obtain permission from the students to use their final grades in our analysis. This was something that we had not obtained for the set of data analyzed here, and thus were unable to factor in course grades. Additionally, we would like to see if students who are living together in on-campus living learning communities are performing similarly. We would need to collaborate with others on campus to acquire this information. We would also like to know if a student's major has any effect on PCE performance. Although we were able to gather some data about majors, much of the data was inaccessible. Perhaps a student survey at the beginning of the semester would be the best way to obtain this information. Since we do have dynamic data, it would be interesting to view student performance over time to see the paths the students are taking toward success on PCE's.

As far as the program itself, policy changes may improve student learning. In the Fall of 2013, we imposed a policy where failure to pass at least 7 competencies resulted in a maximum course grade of D+. In addition to a policy such as this,

it may be beneficial to the students if we require them to pass particular PCE's before each exam. This approach may give students a better idea of where they are with the material and would allow them to drop the course before the deadline, if necessary. Examining student success on individual questions rather than each topic as a whole may also provide insight on ways to improve the program.

We would also like to continue to examine the effects of centrality on a network created from this data. In particular, we are interested in Stephenson and Zelen's information centrality [15]. This measure of centrality tries to capture the information that can be transmitted through all paths between two nodes. The information centrality between nodes i and j is given by $I_{ij} = \frac{1}{g_{ii} + g_{jj} - 2g_{ij}}$ with $g_{ij} \in \mathcal{L}^{-1}(G)$. Since the number of spanning trees in a network represents the efficiency of that network, we are concerned with adding one edge to a network in order to increase $\kappa(G)$ in a maximal way. This is done by adding an edge between the pair i, j with the smallest I_{ij} . We can see why this is true by using Equation 3 in Chapter 1 since

$$I_{ij} = \frac{\kappa(G)}{\kappa(G/e)} = \frac{\kappa(G)}{\partial_{w_{ij}} \kappa(G)}$$

where $\partial_{w_{ij}} \kappa(G)$ is the partial derivative with respect to the weight of the edge, of the complexity polynomial of the network. Hence, I_{ij} is smallest when $\partial_{w_{ij}} \kappa(G)$ is largest, which is precisely when the growth rate is largest [16]. Using this idea, we may be able to further examine how to improve our program.

The techniques used in creating and administering the PCE program would transfer easily to other programs. On a larger scale, an institution could track their students through the typical calculus sequence from precalculus through each semester of calculus by giving a similar exam in each course and tracking student progress. Using a graph would be particularly useful in a situation like this, where the data set is large, and may give the researcher a better idea of how their students

are performing. If a program had the proper flexibility, a tool such as the PCE could be used as a side-by-side course to strengthen knowledge and study skills. A model such as this would allow the students to have more direct instruction with the prerequisite material.

An assessment program such as the PCE would also be a useful way for an instructor to see what the students know coming into a class. This might be particularly useful in a new course or an online course, where the instructor may not know the level of students. Giving an exam similar to the PCE's would allow the instructor to decide what concepts to focus on and which ones the majority of the students already understand.

The administration of the program, data collection, and data analysis provided us with a positive learning experience on remediation. The methods used here are applicable to a wide variety of assessment situations, and a combination of straightforward data analysis and graph modeling can be particularly revealing. The lessons learned in this study will lead to future improvements to this program and other assessment tools.

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APPENDIX A
PCE Exam Spring 2013

This first appendix contains the original exam given to the students in the second week of class.

GENERAL INSTRUCTIONS: Read all instructions carefully.

1. You have 50 minutes to complete the PCE
2. Early departure is authorized. Give your PCE to your instructor when completed.
3. This exam evaluates the understanding of mathematical concepts which are fundamental to students' success in MTH 141. This is a non-technology exam. No references of any kind may be used.
4. Including this cover page, there are 7 pages to the exam.
5. Show all your work. Each problem will be graded either completely right or wrong, but showing your work will allow your instructor to provide you feedback.
6. Place your name on every page.

Name:

Section:

PCE

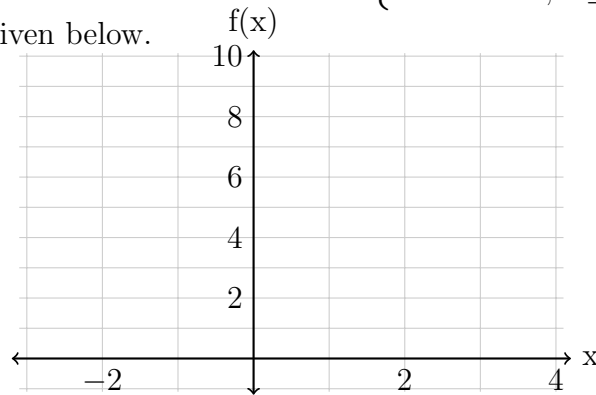
1. Find the domain of $\frac{\sqrt{x-3}}{3x-15}$.

2. Given $f(y) = -2y - 5$ and $g(x) = x - c$, find $f(x+h) - 4g(1)$.

3. Expand and simplify $(x - 5y)(2x - y + 7)$.

4. Factor completely $x^2 + x - 20$.

5. Sketch the graph of the function $Q(x) = \begin{cases} x^2 & ; x > -1 \\ -2x - 1 & ; x \leq -1 \end{cases}$ on the coordinate axes given below.



6. Again, consider the function $Q(x) = \begin{cases} x^2 & ; x > -1 \\ -2x - 1 & ; x \leq -1 \end{cases}$. On what interval is $Q(x)$ decreasing?

7. Simplify $\left(\frac{-3a^{\frac{4}{3}}b^{-2}}{a^2b^{\frac{1}{3}}}\right)^3$ completely, and write your answer without negative exponents.
8. Transform $\frac{-4y}{\sqrt[7]{y-9}^5}$ from rational and radical form to exponential form (fractions are only permitted in the exponent of your answer).
9. Write the equation of the line in slope intercept form passing through the points $(-2, 6)$ and $(1, 0)$.
10. Write the equation in slope intercept form for the line that passes through $(0, 3)$ and is parallel to the line $x - 3y = 7$.
11. Evaluate $\log_3(27)$.
12. Solve $6^{x+3} = 5$ for x . You may leave your answer unsimplified (no calculations required).
13. If possible, solve $-3rs + 2tr = rs^3 - 7$ for r . If it is not possible, briefly explain why.

14. Solve $x^2 - 7x - 5 = -2$ for all possible values of x . You do not need to simplify your answer.
15. Solve $x^2 - 6x + 8 \leq 0$. Give your answer in proper interval notation.
16. Solve $\frac{2x - 8}{x - 1} \geq 0$.
17. Evaluate $\cos(\frac{-\pi}{4})$.
18. Find the range of $f(x) = -2\sin(4x) - 5$.
19. Simplify $-4(11 - 6) + 4^2 \div 8$.
20. How far from the base of a house do you need to place a 13-foot ladder so that it exactly reaches the top of a 12-foot tall wall?

APPENDIX B

One Week of Mini Tests

Each week, a new set of 10 mini tests (one for each competency) is given to the students who still need to master competencies. This appendix shows one week of these tests.

MTH 141: PCE Mini-test

Competency 1, Functions

Name: _____ Section: _____ Date: _____

1. Find the domain of $\frac{\sqrt{x-4}}{5x}$.

2. Given $f(x) = -4x^2$ and $g(x) = -x + 7y$, find $f(y+h) - g(1)$.

MTH 141: PCE Mini-test

Competency 2, Factoring and Expanding

Name: _____ Section: _____ Date: _____

1. Expand and simplify $(10x - 2)(x - 8)$.

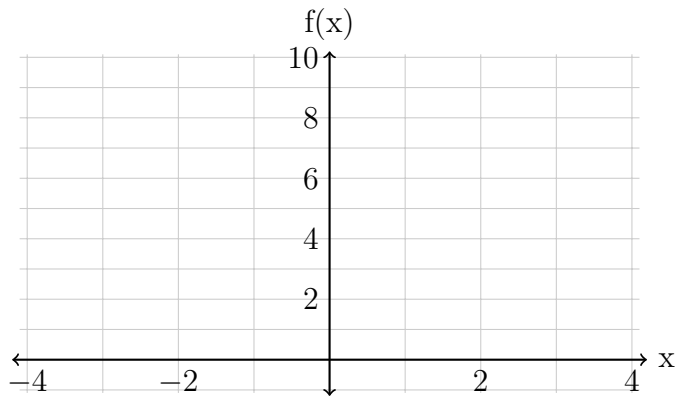
2. Factor $x^4 - 49x^2$ completely.

MTH 141: PCE Mini-test

Competency 3, Graphing and Quadratics

Name: _____ Section: _____ Date: _____

1. Draw the graph of the function $f(x) = x^2 + 4x + 3$ on the coordinate axes given below.



2. Give the interval(s) on which $f(x)$ is negative.

MTH 141: PCE Mini-test

Competency 4, Radicals and Exponents

Name: _____ Section: _____ Date: _____

1. Simplify $\frac{15m^3n^{-5}}{45m^{-1}n^{-3}}$. Write your answer using only positive exponents.

2. Write $\frac{1}{\sqrt[7]{(3xy-9)^2}}$ in exponent form (no radicals permitted).

MTH 141: PCE Mini-test
Competency 5, Straight Lines

Name: _____ Section: _____ Date: _____

1. Write the equation in slope intercept form for the line passing through $(1, 0)$ and $(4, 2)$

2. Write the equation in slope intercept form for the line parallel to $y - 5x = 9$ through the point $(0, -7)$.

MTH 141: PCE Mini-test
Competency 6, Logarithms

Name: _____ Section: _____ Date: _____

1. Write as a single logarithm by using log rules, and simplify: $3\log_7(2y) - \log_7(4z)$

2. Solve $8 = 3^{2x-4}$ for x . (No calculations required - you may leave your answer in log form.)

MTH 141: PCE Mini-test

Competency 7, Algebra

Name: _____ Section: _____ Date: _____

1. Solve $4y^3x^2\frac{dy}{dx} + 2xy^4 - 3y^2\frac{dy}{dx} = 2x$ for $\frac{dy}{dx}$ in terms of all other variables. If it is not possible, briefly state why.

2. Solve $2y^2xz + xy^5 + 10y^2z = 2x - 7y$ for x in terms of all other variables. If it is not possible, briefly state why.

MTH 141: PCE Mini-test

Competency 8, Inequalities

Name: _____ Section: _____ Date: _____

1. Solve $x^2 - 7x + 12 \leq 2$.

2. Solve $\frac{3x - 1}{2x + 4} \geq 1$.

MTH 141: PCE Mini-test
Competency 9, Trigonometric Functions

Name: _____ Section: _____ Date: _____

1. Evaluate $\sin(\pi)$.

2. Evaluate $\cos(\pi)$.

MTH 141: PCE Mini-test
Competency 10, Real Numbers

Name: _____ Section: _____ Date: _____

1. Simplify $-2(7) + 30 \div 5 \times 2 - 4^2$.

2. An increasingly popular way to move these days is to rent a “pod” that you pack yourself. Suppose a particular pod’s walls are each $\frac{1}{2}ft$ thick, and the exterior dimensions of this pod are $9ft \times 10ft \times 11ft$. What is the interior volume of the pod? Be sure to include the correct **units** in your answer. (Hint: It may help to draw some pictures.)

APPENDIX C
Proposed Exam C

This is a proposed mini test C to replace mini tests 2,5,6, and 10.

MTH 141
PCE Mini-test Exam C

Name: _____ Date: _____

1. Factor $x^2 + 8x - 33$ completely.

2. Suppose $f(x)$ is a linear function with slope -4 which passes through the point $(2, 7)$. Find $f(-3)$.

3. Write the equation in slope intercept form for the line perpendicular to $2y + 10x = 1$ through the point $(0, 1)$.

4. Solve $2^{3+\frac{q}{2}} = 6$ for q . (No calculations required - you may leave your answer in log form.)

5. The volume of a sphere is given by $V = \frac{4}{3}\pi r^3$ where r is the radius of the sphere. If the radius is doubled, what happens to the volume? (Hint: It may help to sketch some pictures.)

- (a) It doubles.
- (b) It is halved.
- (c) It stays the same.
- (d) It is eight times bigger.
- (e) It is four times bigger.
- (f) None of the above.

APPENDIX D

Final Exam

We re-tested each competency on the final exam. These ten questions from the final exam are shown below.

Final Exam

Spring 2013

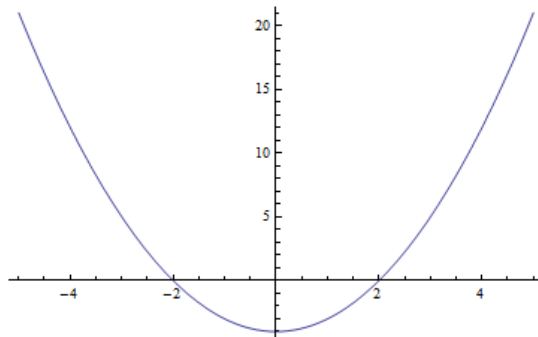
1. If $f(x) = x^2$ and $g(x) = x - k$, find $f(-3) - g(4)$.

- (A) $5 + k$
- (B) $-13 + k$
- (C) $5 - k$
- (D) $-13 - k$

2. Which of the following gives a complete list of the roots of $x^3 - 9x$?

- (A) $0, 3, -3$
- (B) $0, 3$
- (C) $-3, 3$
- (D) 3

3. The graph below is most likely the graph of which of the following functions?



(A) $f(x) = x^2 + 4$

(B) $f(x) = -x^2 - 4$

(C) $f(x) = -x^2 + 4$

(D) $f(x) = x^2 - 4$

4. Simplify completely: $\left(\frac{-3a^{1/3}b}{a^2b^{-1/3}}\right)^3$.

(A) $\frac{-3b^4}{a^5}$

(B) $\frac{27b^4}{a^5}$

(C) $\frac{b^2}{27a^5}$

(D) $\frac{-27b^4}{a^5}$

5. Write the equation of the line perpendicular to $2y - x = -7$ that goes through the point $(0, 4)$.

(A) $y = -2x + 4$

(B) $y = -2x - \frac{7}{2}$

(C) $y = \frac{1}{2}x - \frac{7}{2}$

(D) $y = \frac{1}{2}x + 4$

6. Evaluate: $\log(1,000,000)$.

(A) $(10)^{1,000,000}$

(B) 7

(C) 100,000

(D) 6

7. If possible, solve the equation below for w : $w \cos(x) + x^2 = \frac{1}{3}w + we^x$.

(A) $\frac{\cos(x) - \frac{1}{3} - e^x}{-x^2}$

(B) $\frac{x^2}{\cos(x) + \frac{1}{3} + e^x}$

(C) $\frac{-x^2}{\cos(x) - \frac{1}{3} - e^x}$

(D) It is not possible to solve for w .

8. Solve $x^2 + 2x - 35 \leq 0$.

- (A) $[-7, 5]$
- (B) $x = -7, x = 5$
- (C) $(-\infty, -7] \cup [5, \infty)$
- (D) $(-\infty, -7]$

9. Which of the following are true?

- | | |
|--------------------|---------------------|
| (I) $\sin(0) = 0$ | (III) $\cos(0) = 0$ |
| (II) $\sin(0) = 1$ | (IV) $\cos(0) = 1$ |

- (A) II and III
- (B) I and IV
- (C) II and IV
- (D) I and III

10. The gravitational force, F , between two objects is given by $F = \frac{Gm_1m_2}{r^2}$, where, G is the universal gravitational constant, m_1 and m_2 are the masses of the two objects, and r is the distance between the objects. If r is increased while G , m_1 , and m_2 remain constant, then what happens to the gravitational force?

- (A) F increases.
- (B) F decreases.
- (C) F stays the same.
- (D) There is not enough information to determine how or if the gravitational force will change.

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